VECTOR CALCULUS

Q.

Prove that

$$\vec{F} = (2xy^2 + yz)\vec{i} + (2x^2y + xz + 2yz^2)\vec{j} + (2y^2z + xy)\vec{k}$$ is a conservative field.

Soln

For a conservative field, curl $\vec{F} = \vec{0}$.

Now

$$\text{curl } \vec{F} = \nabla \times \vec{F}$$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \times \vec{F}$$

$$\Rightarrow \text{curl } \vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \times \left[(2xy^2 + yz)\vec{i} + (2x^2y + xz + 2yz^2)\vec{j} + (2y^2z + xy)\vec{k}\right]$$

$$\Rightarrow \text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy^2 + yz & 2x^2y + xz + 2yz^2 & 2y^2z + xy \end{vmatrix}$$

$$\Rightarrow \text{curl } \vec{F} = \vec{i}\left[\frac{\partial}{\partial y}(2y^2z + xy) - \frac{\partial}{\partial z}(2x^2y + xz + 2yz^2)\right]$$

$$- \vec{j}\left[\frac{\partial}{\partial x}(2y^2z + xy) - \frac{\partial}{\partial z}(2xy^2 + yz)\right]$$

$$+ \vec{k}\left[\frac{\partial}{\partial x}(2x^2y + xz + 2yz^2) - \frac{\partial}{\partial y}(2xy^2 + yz)\right]$$

$$\Rightarrow \text{curl } \vec{F} = \vec{i}\left[\cancel{4yz} + \cancel{x} - \cancel{x} - \cancel{4yz}\right]$$

$$- \vec{j}\left[\cancel{y} - \cancel{y}\right] + \vec{k}\left[4\cancel{xy} + \cancel{z} - 4\cancel{xy} - \cancel{z}\right]$$

$$= \vec{0}$$

Thus, $\text{curl } \vec{F} = \vec{0}$.

$\Rightarrow \vec{F}$ is a conservative field.